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BOUNDARY LAYER CONDITIONS IN FREE CONVECTION

by

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1. *Introduction*

The great success of the investigations into flow of viscous fluids along solid surfaces has to be contributed primarily to the concept of the boundary layer as proposed by Prandtl [1]. It is well-known to those familiar with the complexity of the Navier-Stokes equations that the boundary layer approximations are a powerful means of reducing these equations to proportions which allow them to be solved. At the same time, however, the introduction of the boundary layer approximations is accompanied by conditions restricting the range of application. For example, taking the case of forced flow along solid bodies, Prandtl derived that $Re^{\frac{1}{2}} >> 1$ has to be satisfied for the boundary layer approximations to be valid.

In free convection the idea has persisted up to now that the condition $Gr^{\frac{1}{4}} >> 1$ has to be satisfied if the solutions are to be drawn from boundary layer equations. It is the purpose of this paper to show that this condition is not valid for small or large Prandtl numbers. Therefore it has been judged necessary to carefully derive the restricting conditions. A great asset of this investigation is that it shows the way to the most natural method of solving the free convection boundary layer equations under extreme Prandtl number conditions.

2. *Equations*

The mathematical model to be used is shown in figure 1.

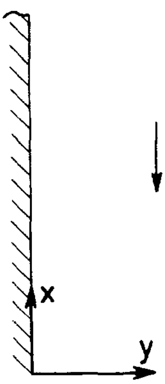

Fig. 1.

If we restrict ourselves in the analysis to ordinary liquids, the equation of state giving the relation between the density and the temperature will

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be*

$$\rho = \rho_\infty \left\{ 1 - \beta(T-T_\infty) \right\}. \tag{1}$$

This naturally is the linear approximation to a more intricate equation of state. It may be used if

$$\epsilon = \beta(T_w - T_\infty) << 1. \tag{2}$$

For simplicity we assume T_w constant in the present analysis. Apart from the equation of state there are four other equations governing the phenomenon of free convection in steady state, viz., the equation of continuity

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0, \tag{3}$$

the momentum equations

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\left\{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right\} - \frac{1}{\rho}\frac{\partial p}{\partial x} - g, \tag{4}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = \nu\left\{\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right\} - \frac{1}{\rho}\frac{\partial p}{\partial y}, \tag{5}$$

the energy equation

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{\lambda}{\rho c_p}\left\{\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right\} + \frac{\beta T}{\rho c_p}\left\{u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y}\right\} +$$
$$+ \frac{\nu}{c_p}\left[2\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right\} + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2 - \frac{2}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)^2\right]. \tag{6}$$

Here the last term of equation (6) is the dissipation function.

The task which now lies before us is to determine the conditions under which the Navier-Stokes equations and the energy equation may be reduced to their boundary layer approximations. Ostrach [2] in particular, has performed a great deal of work to achieve this aim. The derivation to be given here differs in many respects from that of Ostrach who states that for the Grashof number

$$Gr = \frac{g\beta(T_w - T_\infty)\ell^3}{\nu^2} \tag{7}$$

large enough the second order derivatives with respect to x may be neglected in comparison with those taken with respect to y. This would mean that for an inviscid fluid ($\nu = 0$) there would always be a boundary layer, which is certainly not true. We anticipate that for small viscosities Ostrach's condition $Gr^{\frac{1}{4}} >> 1$ has to be replaced by a condition not involving the viscosity. In deriving the restricting conditions one has probably been influenced too much by the achievements of forced laminar boundary layer flow. A desire for parallelism made earlier investigators decide to impose the condition that the viscous-and the inertia terms had to be of the same order of magnitude in the free convection boundary layer. This is in close

* For Nomenclature, see p. 104.

analogy with the earlier findings of forced laminar boundary layer flow. As a consequence the Prandtl number

$$\sigma = \frac{\nu \rho c_p}{\lambda} \tag{8}$$

representing the connection between the viscous-and the temperature boundary layer appeared in the energy equation. It is, however, a known fact that the answer to the question as to which terms have to be of the same order of magnitude in the equations of free convection depends primarily on the value of the Prandtl number. So rather than trying to find one single expression on which to decide the applicability of the boundary layer approximations we have to expect different criteria for different Prandtl numbers. Although the boundary layer equations ultimately found by Ostrach are correct the derivation leads in some cases to dubious conclusions about the conditions which, on being satisfied, permit the boundary layer approximations. As an example we may give, that instead of $Gr^{\frac{1}{4}} \gg 1$ we have to put $\sigma\, Gr^{\frac{1}{4}} \gg 1$ as the condition justifying the boundary layer approximations at low Prandtl numbers (nearly inviscid fluid). This will be proved later on.

This last condition is found by considering that for $\sigma < 1$ the temperature boundary layer is always somewhat thicker than the velocity boundary layer. Moreover, the viscous effects decrease with σ. That is why for $\sigma < 1$ the boundary layer equations will be derived through stating that for small Prandtl numbers the convection- and conduction terms of the energy equation have to be of the same order of magnitude. In the equations obtained the Prandtl number will be the coefficient of the viscous term thus asserting that for $\sigma \to 0$ the viscous stresses are of minor importance.

It may be clear from these statements that there is still need for an investigation, having the character of a scrutiny, concerning the conditions permitting the boundary layer approximations.

3. *Non-dimensional variables*

Let us introduce non-dimensional variables -indicated with bars- as follows

$$\begin{aligned} x &= \ell\bar{x}, \qquad & y &= \delta\ell\bar{y}, \qquad & u &= U\bar{u}, \\ \rho &= \rho_\infty \bar{\rho}, \qquad & p &= \rho_\infty U^2\bar{p}, \qquad & T &= T_\infty + (T_w - T_\infty)\bar{T}. \end{aligned} \tag{9}$$

δ, ℓ and U will be chosen later on in such a way as to make $\bar{x}$, $\bar{y}$ and $\bar{u}$ of order unity in the boundary layer. Apparently the equation of state (1) will become

$$\bar{\rho} = 1 - \epsilon\bar{T}. \tag{10}$$

Bearing in mind that $\delta\ell$ is the thickness of the boundary layer an integration of equation (3) across the boundary layer gives

$$v = - U\delta \left\{ \frac{1}{\bar{\rho}} \int_0^1 \frac{\partial}{\partial \bar{x}} (\bar{\rho}\bar{u}) d\bar{y} \right\}. \tag{11}$$

Obviously $v = 0(U\delta)$ so that in addition to (9) we may introduce

$$v = \delta U \bar{v}. \tag{12}$$

Substituting (9), (10) and (12) in (3), (4), (5) and (6) gives

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} - \frac{\epsilon}{1-\epsilon\bar{T}} \left\{ \bar{u}\, \frac{\partial \bar{T}}{\partial \bar{x}} + \bar{v}\, \frac{\partial \bar{T}}{\partial \bar{y}} \right\} = 0, \tag{13}$$

$$\bar{u}\, \frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\, \frac{\partial \bar{u}}{\partial \bar{y}} = \frac{\nu}{U\ell}\, \frac{1}{\delta^2} \left\{ \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} + \delta^2\, \frac{\partial^2 \bar{u}}{\partial \bar{x}^2} \right\} +$$

$$- \frac{1}{1-\epsilon\bar{T}}\, \frac{\partial \bar{p}}{\partial x} - \frac{g\ell}{U^2}, \tag{14}$$

$$\bar{u}\, \frac{\partial \bar{v}}{\partial \bar{x}} + \bar{v}\, \frac{\partial \bar{v}}{\partial \bar{y}} = \frac{\nu}{U\ell}\, \frac{1}{\delta^2} \left\{ \frac{\partial^2 \bar{v}}{\partial \bar{y}^2} + \delta^2\, \frac{\partial^2 \bar{v}}{\partial \bar{x}^2} \right\} +$$

$$- \frac{1}{1-\epsilon\bar{T}}\, \frac{1}{\delta^2}\, \frac{\partial \bar{p}}{\partial \bar{y}}, \tag{15}$$

$$\bar{u}\, \frac{\partial \bar{T}}{\partial \bar{x}} + \bar{v}\, \frac{\partial \bar{T}}{\partial \bar{y}} = \frac{\lambda}{\rho_\infty c_p \ell U}\, \frac{1}{\delta^2}\, \frac{1}{1-\epsilon\bar{T}} \left\{ \frac{\partial^2 \bar{T}}{\partial \bar{y}^2} + \delta^2\, \frac{\partial^2 \bar{T}}{\partial \bar{x}^2} \right\} +$$

$$+ \frac{\beta U^2}{c_p(T_w - T_\infty)}\, \frac{T_\infty + (T_w - T_\infty)\bar{T}}{1-\epsilon\bar{T}} \left\{ \bar{u}\, \frac{\partial \bar{p}}{\partial \bar{x}} + v\, \frac{\partial \bar{p}}{\partial \bar{y}} \right\} +$$

$$+ \frac{\nu U}{c_p \ell (T_w - T_\infty)}\, \frac{1}{\delta^2}\, \frac{1}{1-\epsilon\bar{T}} \left[2 \left\{ \left(\frac{\partial \bar{u}}{\partial \bar{x}}\right)^2 + \left(\frac{\partial \bar{v}}{\partial \bar{y}}\right)^2 \right\} \delta^2 + \right.$$

$$\left. + \left(\frac{\partial \bar{u}}{\partial \bar{y}} + \delta^2\, \frac{\partial \bar{v}}{\partial \bar{x}} \right)^2 - \frac{2}{3} \left(\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} \right)^2 \delta^2 \right]. \tag{16}$$

Let us first direct our attention to equation (15). With respect to the orders of magnitude of the inertia- and the viscous terms we distinguish three different cases. The inertia terms can be larger, equal or smaller than the viscous terms. For the former two cases it is clear that the boundarey layer approximation to equation (15) is

$$\frac{\partial \bar{p}}{\partial \bar{y}} = O(\delta^2)$$

Since δ is supposed to be small ($\delta^2 << 1$) it is obvious that equation (15) reduces to

$$\frac{\partial \bar{p}}{\partial \bar{y}} = 0 \tag{17}$$

If, however, the viscous terms are larger than the inertia terms we have

$$\frac{\partial \bar{p}}{\partial \bar{y}} = O\left(\frac{\nu}{U\ell}\right)$$

Upon imposing the condition

$$\frac{\nu}{U\ell} << 1 \tag{18}$$

for this special case it follows that equation (17) can still be used as the the boundary layer approximation to equation (15).
Turning to equation (14) we see that it will reduce to

$$\frac{\partial \bar{p}}{\partial \bar{x}} = -\frac{g\ell}{U^2} \tag{19}$$

as $\bar{y} \to \infty$. This follows directly from boundary conditions $\bar{u} \to 0$, $\bar{T} \to 0$ as $\bar{y} \to \infty$. On account of equation (17) the expression (19) is also valid in the boundary layer. Insertion of (19) in (14) yields

$$\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = \left\{ \frac{\nu}{U\ell}\frac{1}{\delta^2}\frac{\partial^2 \bar{u}}{\partial \bar{y}^2} + \frac{\epsilon g \ell}{U^2}\bar{T} \right\}\left\{ 1 + O(\epsilon) \right\} . \tag{20}$$

Here we have already made use of $\delta^2 \ll 1$. Under the same condition equation (16) will become

$$\bar{u}\frac{\partial \bar{T}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{T}}{\partial \bar{y}} = \left\{ \frac{\lambda}{\rho_\infty c_p \ell U}\frac{1}{\delta^2}\frac{\partial^2 \bar{T}}{\partial \bar{y}^2} - \frac{\beta g \ell \left\{ T_\infty + (T_w - T_\infty)\bar{T} \right\}}{c_p (T_w - T_\infty)}\bar{u} + \right.$$

$$\left. + \frac{\nu U}{c_p \ell (T_w - T_\infty)}\frac{1}{\delta^2}\left(\frac{\partial \bar{u}}{\partial \bar{y}}\right)^2 \right\}\left\{ 1 + O(\epsilon) \right\} , \tag{21}$$

while for the equation of continuity we find

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = O(\epsilon) \tag{22}$$

We have now arrived at the important question as to which conditions should determine δ. If we decide that after dropping the $O(\epsilon)$ terms every term of equation (20) has to be of the same order of magnitude, we would obtain the traditional results. It is felt, however, that in this way too much emphasis is put on the momentum equation. In free convection the energy equation should also receive adequate attention since it is the temperature-differences which lie at the very root of the phenomenon. The known results of free convection suggest that our investigation be split up in two different studies. One should be concerned with small Prandtl numbers ($\sigma < 1$), the other should refer to large Pradntl numbers ($\sigma > 1$).

4. *Small Prandtl numbers*

First we investigate low Prandtl numbers, i.e. nearly inviscid fluids. We impose the condition that the convection- and conduction terms of equation (21) be of the same order of magnitude. Hence

$$\delta^2 = \frac{\lambda}{\rho_\infty c_p \ell U} \tag{23}$$

Since in free convection the sole driving force is represented by the buoyancy term this term must be of the same order of magnitude as the largest terms of equation (20). In the present case these obviously are the inertia terms. We consequently find

$$U^2 = \epsilon g \ell = \beta g \ell (T_w - T_\infty) \tag{24}$$

The combined knowledge of (23) and (24) yields

$$\delta^2 = \sigma^{-1} Gr^{-\frac{1}{2}} \tag{25}$$

Obviously for small Prandtl numbers the condition

$$\sigma\, Gr^{\frac{1}{2}} >> 1 \tag{26}$$

has to be satisfied for the boundary layer approximations to be valid. Since c_p is usually very large and β very small the terms

$$\frac{\beta g \ell T_\infty}{c_p(T_w - T_\infty)} \quad ; \quad \frac{\nu U}{c_p \ell (T_w - T_\infty)} \frac{1}{\delta^2} = \sigma \frac{g\beta\ell}{c_p} \tag{27}$$

are small. Hence the two last terms of equation (21) may be neglected. On finally imposing the condition $\epsilon << 1$ (2) we find for small Prandtl numbers the set of governing boundary layer equations

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0, \tag{28}$$

$$\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = \sigma \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} + \bar{T}, \tag{29}$$

$$\bar{u}\frac{\partial \bar{T}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{T}}{\partial \bar{y}} = \frac{\partial^2 \bar{T}}{\partial \bar{y}^2}. \tag{30}$$

It may be noted that the Prandtl number occupies an important position in determining the influence of the viscous stresses in a basically inviscid fluid. Since in the analysis of Ostrach the viscous stresses are judged to be of the same order of magnitude as the inertia terms, his analysis evidently applies to the viscous part of the free convection boundary layer which is known to approach zero as $\sigma \to 0$. So his analysis can give interesting information about the ratio of the thickness of the viscous layer δ_v and the thickness of the full free convection boundary layer. On replacing δ in (9) and (12) by δ_v and making all terms in the momentum equation of equal order of magnitude we readily derive

$$\delta_v = Gr^{-\frac{1}{4}}. \tag{31}$$

With (25) this clearly gives

$$\delta_v / \delta = \sigma^{\frac{1}{2}}. \tag{32}$$

The boundary layer momentum- and energy equation now obviously are given by

$$\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} + \bar{T}, \tag{33}$$

$$\sigma\left\{\bar{u}\frac{\partial \bar{T}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{T}}{\partial \bar{y}}\right\} = \frac{\partial^2 \bar{T}}{\partial \bar{y}^2}, \tag{34}$$

which is, as far as the position σ occupies is concerned, the traditional way of presenting the boundary layer equations of free convection. The problem of small Prandtl number free convection has been solved explicitly for the isothermal flat plate by means of the method of matched asymptotic expansions [3]. We accordingly may refer to this work for further information.

5. *Large Prandtl numbers*

As we have remarked earlier the thermal- and the velocity boundary layer are of about the same thickness for $\sigma < 1$, the former being somewhat thicker than the latter. Concerning this we may refer to [3] or to the work of Sparrow & Gregg [4] on low Prandtl numbers. This is contrary to the situation in forced flows where the velocity layer is drowned in the thermal layer as $\sigma \to 0$. In free convection an increasing part (as $\sigma \to 0$) of the velocity layer is inviscid so that the Prandtl number cannot supply information about the relation between the thermal- and the complete velocity boundary layer. For $\sigma > 1$, however, the physical pattern reflects the same features as in forced convection, i.e. the thermal boundary layer is thin in comparison with the velocity boundary layer. The velocity boundary layer is totally viscous. If we fix our attention now on very large values of σ ($\sigma >> 1$) the following picture emerges (see [2] or [5]). Let us consider a fluid of large viscosity and small thermal conductivity. For such a fluid the Prandtl number is large. Now, obviously, the temperature boundary layer will be very thin thus only admitting buoyancy forces in this very thin layer. In this layer the fluid will be dragged upward. Due to the large viscosity the fluid will also move upwards in an adjacent layer of considerable thickness where no buoyancy forces exist. We obviously have to use the following model in deriving the boundary layer equations and in stating the conditions of their applicability. In the thin thermal boundary layer the convection terms and the conduction terms are of the same order of magnitude. In the momentum equation the buoyancy term and the viscous term have to be of the same order of magnitude. Making use of these considerations we may find the thickness of the thermal boundary layer and, what is very important, a characteristic velocity. Since in the layer where no buoyancy forces are present the flow can only be retarded this velocity must also be the characteristic velocity of the complete viscous layer. Using this velocity and the condition that in the viscous layer the inertia- and the viscous terms are of the same order of magnitude we can derive an expression for the thickness of the free convection boundary layer at large Prandtl number. Here the outer fringes of the viscous layer determine this thickness.

Now fixing our attention first on the thermal layer, the condition of the conduction- and convection terms being of comparable magnitude leads to (see equ. (21))

$$\delta_T^2 = \frac{\lambda}{\rho_\infty c_p \ell U} \tag{35}$$

The suffix T naturally refers to the fact that δ_T is not the thickness of the complete free convection boundary layer but only of that part of it where tangible temperature-differences with the ambient fluid exist. Our condition that the buoyancy term matches with the viscous term leads to (equ. (20))

$$U = \frac{g\beta\ell^2(T_w - T_\infty)}{\nu}\,\delta_T^2 \tag{36}$$

Substitution of (35) in (36) renders

$$U^2 = \frac{g\beta\ell(T_w - T_\infty)}{\sigma} \tag{37}$$

and

$$\delta_T^2 = \sigma^{-\frac{1}{2}}\ Gr^{-\frac{1}{2}}\ . \tag{38}$$

The fact, as expressed by (37), that the velocity decreases as σ increases is in complete agreement with earlier findings [2]. As has been remarked above, the character of a free convection boundary layer at large Prandtl number is one of a viscous layer in forced flow, the force being exerted through buoyancy in a very thin layer adjacent to the wall. As a consequence we can only impose one condition for determining the thickness of the layer. This condition naturally is the same as the one used by Prandtl [1] in discussing a viscous boundary layer of forced flow: it expresses, that in the layer the inertia- and the viscous terms are of the same order of magnitude. Using (20) we find

$$\delta^2 = \frac{\nu}{U\ell} \tag{39}$$

For reasons advanced earlier we may use equation (37) as expressing adequately the characteristic velocity in equation (39). This leads to

$$\delta^2 = \sigma^{\frac{1}{2}}\,Gr^{-\frac{1}{2}} \tag{40}$$

As the condition $\delta^2 \ll 1$ coincides with (18) we have to impose-in case σ is large-

$$\sigma^{-\frac{1}{2}} Gr^{\frac{1}{2}} \gg 1 \tag{41}$$

for the boundary layer approximations to be valid. Another interesting outcome of the present analysis is that it supplies evidence about both the thermal- and the viscous layer. Using (38) and (40) we find

$$\delta_T / \delta = \sigma^{-\frac{1}{2}} \tag{42}$$

The figures of Ostrach [2] about free convection at large Prandtl numbers confirm qualitatively expression (42). As one is left with a certain amount of uncertainty in choosing the outer edge of a boundary layer the = sign could be replaced best by a $\sim$ sign. Bearing in mind that for $\sigma < 1$ the thermal boundary layer is predominant, while for $\sigma > 1$ this is the case with the viscous layer, both formulas (32) and (42) lead to

$$\delta_v / \delta_T \sim \sigma^{\frac{1}{2}}. \tag{43}$$

After having thrown light upon the different aspects of large Prandtl number free convection boundary layer flow, it may have become clear that the only way to solve it realistically is be using the method of matched inner- and outer expansions. The inner problem can be studied by working in the exiguous dimensions of the thermal boundary layer. On replacing δ in (9) and (12) by δ_T the substitution of (37) and (38) in (20) and (21) then leads to the following momentum- and energy equation

$$\bar{u}\,\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\,\frac{\partial \bar{u}}{\partial \bar{y}} = \sigma \left\{ \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} + \bar{T} \right\}, \tag{44}$$

$$\bar{u}\,\frac{\partial \bar{T}}{\partial \bar{x}} + \bar{v}\,\frac{\partial \bar{T}}{\partial \bar{y}} = \frac{\partial^2 \bar{T}}{\partial \bar{y}^2} \tag{45}$$

In solving these equations one has to use the inner boundary conditions (conditions at the wall). The remaining conditions for $\bar{y} \to \infty$ have to be found through matching with the solution of the outer problem.

Inserting the expressions (37) and (39) in the equations (20) and (21) we scale up to the larger dimensions of the complete viscous layer. The equations become

$$\bar{u}\,\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\,\frac{\partial \bar{u}}{\partial \bar{y}} = \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} + \sigma\,\bar{T}, \tag{46}$$

$$\sigma \left\{ \bar{u}\,\frac{\partial \bar{T}}{\partial \bar{x}} + \bar{v}\,\frac{\partial \bar{T}}{\partial \bar{y}} \right\} = \frac{\partial^2 \bar{T}}{\partial \bar{y}^2} \tag{47}$$

At first sight it seems rather contradictory that the buoyancy term in the momentum equation contains the large parameter σ. We should, however, remember that for the main term of the outer expansion the temperature is exactly equal to zero. As a consequence the buoyancy term plays, as expected, no part in the main term of the outer expansion. This main term is a solution of the differential equation

$$\bar{u}\,\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\,\frac{\partial \bar{u}}{\partial \bar{y}} = \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} \tag{48}$$

and has to satisfy the outer boundary conditions $\bar{u} \to 0$ as $\bar{y} \to \infty$. The remaining inner boundary conditions have to be found through matching with the inner problem according to the well-known matching rule (see [6]).

The author is pursuing further study on this subject by applying matched asymptotic expansions to a large Prandtl number problem.

6. *Conclusions*

Although separate analyses have been performed for extreme values of the Prandtl number it may be expected that the results drawn therefrom are qualitatively consistent for a larger Prandtl number range as long as in this range the basic assumptions remain the same qualitatively. Consequently the results obtained for small Prandtl numbers are expected to give information for $\sigma < 1$, while those found for large Prandtl numbers are believed to be valuable for $\sigma > 1$. Hence for $\sigma < 1$, we have that the boundary layer approximations are valid provided

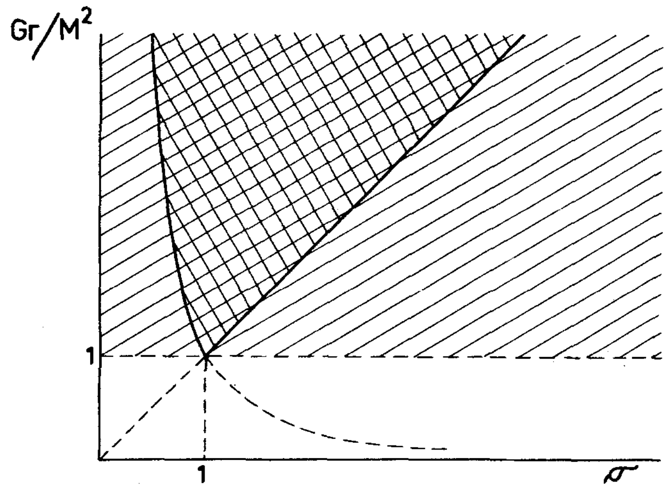

Fig. 2. Conditions for boundary layer approximations. Double shaded: present work; single shaded: Ostrach [2].

$$\sigma\ Gr^{\frac{1}{2}} > M. \tag{49}$$

Here we have rephrased equation (26) through introduction of a very large number M so as to give it a more definite character. For $\sigma > 1$ we find through (41) the analogous condition

$$\sigma^{-\frac{1}{2}}\ Gr^{\frac{1}{2}} > M. \tag{50}$$

Although near $\sigma = 1$ the graph of figure 2 may have to be changed somewhat it clearly exhibits the result of the present analysis. While Ostrach's analysis merely gives $Gr^{\frac{1}{2}} > M$ the present investigations reveal that stricter rules have to be imposed upon the Grashof number if the boundary layer approximations are to be valid. The single shaded region applies to the condition of Ostrach while the double shaded region is a result of the present investigations.

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NOMENCLATURE

c_p	specific heat
g	acceleration due to gravity
Gr	Grashof number (7)
ℓ	characteristic length (9)
M	large number
p	pressure
T	temperature
T_w	wall-temperature
T_∞	ambient temperature
u	longitudinal velocity (x-direction)
U	characteristic velocity (9)
v	normal velocity (y-direction)
x	longitudinal coordinate (along the plate)
y	normal coordinate (normal to the plate)

Greek symbols

β	coefficient of thermal expansion
δ	non-dimensional thickness of the boundary layer (9)
δ_v	non-dimensional thickness of the viscous boundary layer
δ_T	non-dimensional thickness of the thermal boundary layer
ϵ	small coefficient (2)
λ	coefficient of heat conduction
ν	kinematic viscosity
ρ	density of the fluid
ρ_∞	density of the ambient fluid
σ	Prandtl number (8)

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